# Some Optimal Low-Acceleration Rendezvous Maneuvers

B H BILLIK*
*Aerospace Corporation, El Segundo, Calif*

**For each of three specific rendezvous missions, the low-acceleration piogram which optimizes performance, as referred to some figure of merit, is obtained Specifically, these figures of merit are either propellant or time, and each of the missions is peiformed in a manner that will minimize one of these quantities subject to suitable constraints Whenever possible, the differential equations of motion are integrated, with the use of the optimum acceleiation program, to obtain the trajectory of the vehicle(s) as an explicit function of time In general, the equations from which the desired optimal results are to be extracted are sufficiently complicated so that some machine computation is required The emphasis on such computation varies between the three solutions An illustrative example comparing an optimum acceleration program with an "equivalent" constant tangential acceleration program is included in the paper This comparison shows that the optimum program is slightly superior to the tangential program The magnitude and direction of the optimal acceleration vector is shown to vary The optimum low-acceleration programs are obtained by employing the technique of differential games A brief summary of the philosophy involved in this optimization procedure and a list of required equations are included for reference purposes**

## I Introduction

IN the following paper, a trio of optimal low-acceleration rendezvous problems is analyzed by employing the techniques afforded through the application of differential games The underlying theory involved in the formulation and solution of optimal strategy problems via differential games was described in detail by Isaacs in Refs 1–4 Although the procedure is intrinsically identical to that which evolves from the use of the Pontryagin maximum principle, we will nevertheless adhere to the notation and formulation as presented in the former A brief synopsis of the required optimality equations is included for reference purposes

Consider a target satellite traveling in a circular orbit of radius $r_0$ about the earth For the present, it will be presumed that this satellite is devoid of any thrusting capability An interceptor vehicle is assumed to be located at a distance $P(t)$ from the target, where $P(t) \ll r_0$ for all $t$ Conceptually, both satellites are located in nearby orbits and are traveling relatively close to each other The interceptor is equipped with an engine capable of delivering continuous acceleration of variable magnitude Two distinct closure maneuvers are contemplated. These are as follows:

1) To transfer the interceptor from its initial state to a final state, which is coincident with the target in both position and velocity, in a fixed time $T$ This mission is to be performed in a manner that minimizes the required propellant, i e , a minimum energy closure maneuver For brevity, the foregoing will be referred to as passive rendezvous

2) To transfer the interceptor from its initial state to one in which the interceptor passes at a distance $\delta$ from the target, in a specified time $T$ At the termination of this maneuver, the target and interceptor are constrained to be at the same altitude † A minimum energy maneuver is again desired We will denote this mission by the title passive fly-by

It should be noted that, whereas the passive rendezvous maneuver constrains the final state of the interceptor to prescribed values, the fly-by maneuver leaves the terminal velocity completely free and the terminal geometrical position constrained only to lie on a circle of radius $\delta$ centered at the target The possible applications of the former are obvious whereas the latter might well be employed to gather surveillance information on a foreign satellite vehicle Neither of the forementioned missions (theoretically) requires the use of any high-thrust terminal corrections

In the sections to follow, solutions are obtained to both the passive and fly-by rendezvous missions Specifically, the optimal acceleration vector control and interceptor trajectories are obtained as closed form functions of the time and, in addition, expressions are derived which yield the propellant expended Generally speaking, the complexity of these extremal solutions is such that machine computations will invariably be required In particular, the problem of obtaining the constants of integration to be employed in the equations defining the fly-by mission requires the solution of a set of nonlinear algebraic expressions The solutions to these expressions can best be obtained by employing high-speed machine computation

In the final problem analyzed, the game theoretic aspects of the optimization technique used in this report are brought to bear on a rendezvous mission that involves optimal tactics In the context of the mission to be described immediately below, the interceptor and target vehicles may be thought of as the pursuer and evader, respectively This latter notation is in keeping with that used in Refs 1–4

Let us assume the same initial geometrical configuration between the pursuer and evader as outlined for the first two

Received August 27, 1964 This paper was prepared for Commander, Space Systems Division, U S Air Force, Inglewood, Calif , under Contract No AF04(695)-169 The author wishes to thank E Levin (Head, Astrodynamics Department) and H K Karrenberg (Head, Orbit Designation Section) for their aid in the analysis

* Member of the Technical Staff, Astrodynamics Department Member AIAA

† This constraint is included merely to simplify the manipulation of the equations With a little effort, it could be relaxed entirely The constraint as just worded is only approximately true Actually, the target and interceptor are required to lie in a plane tangent to a sphere of radius $r_0$, where $r_0$ is the radial distance to the target

problems For this case, both vehicles will be assumed to have low-acceleration engines, and we will allow the acceleration vectors of either vehicle to be arbitrarily oriented in space while the magnitudes of these vectors are fixed, i e, constant acceleration From the point of view of the pursuer ($P$), a strategy is desired which will permit this vehicle to rendezvous with the evader in the minimum time In contrast to this, the evader desires a strategy that will yield the maximum time befoie rendezvous is affected

Although much is left to be said concerning this mission, we will leave such discussion to the section in which the problem is treated This mission will be referred to as an active rendezvous

A particular example involving the solution to the passive rendezvous problem is treated in Sec D The results, as obtained from the optimal equations, are compared with a constant tangential acceleration program, which is known, a posteriori, to be capable of performing the desired mission

The physical assumptions inherent in the analysis are that the vehicle(s) moves in an inverse square gravitational field and natural perturbations are negligible, i e, oblateness drag, sun, moon, etc

## II Differential Games: Optimality Equations

In this section, a succinct and admittedly cursory exposition is given on the formulation and solution of optimal strategy problems through the use of differential games In essence, it is nothing more than a brief outline of those features of the techniques which are to be employed in the sections to follow The entire topic of differential games is, however, discussed in great detail in Refs 1–4, and the reader is directed there for a complete dissertation For simplicity, the topic will be discussed from the point of view of the motion of two vehicles

Consider two vehicles, the totality of whose descriptive variables, i e, state variables, can be represented by the vector $\mathbf{X} = (x_1, x_2, \quad, x_n, x_{n+1})$, where the $x_i$ are defined in some subspace $E$ of Euclidean $n + 1$ space We will assume, at each instant of time, that the set of such variables associated with a particular vehicle is known by the other Hence, we are concerned with a deterministic problem of complete information Let the dynamical equations governing the changes of these descriptive variables with time be

$$dx_i/dt = f_i(x_1, x_2, \quad, x_n, x_{n+1}, \phi_1, \phi_2, \quad, \phi, \psi_1, \psi_2, \quad, \psi_s) \qquad i = 1,2, \quad, n$$

$$dx_{n+1}/dt = 1 \tag{2 1}$$

or

$$dx_i/dt = f_i(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi}) \qquad i = 1,2, \quad, n+1 \tag{2 2}$$

where the vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are the navigation variables of the vehicles, i e, steering functions ‡ The $f_i$ are assumed continuous with respect to $\mathbf{X}$, $\boldsymbol{\phi}$, and $\boldsymbol{\psi}$ and continuously differentiable with respect to $\mathbf{X}$

In order to classify what is to follow as a game, conflicting objectives will be attributed to the two vehicles That is, one of the vehicles is trying to maximize some quantity while the other is endeavoring to minimize the same quantity For simplicity, the duration of the game will be assumed fixed The game is considered over when some preassigned objective is met Usually, this can be interpreted as implying that $\mathbf{X}$ lies on some specified surface of dimension $d \leq n$ in $E$ We will title this surface the terminal surface and denote it by the letter $C$ In particular, $C$ might be a point in $E$

For prescribed functions $G(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi})$ and $F(\mathbf{X})$ (differentiable with respect to their arguments), an integral payoff $J$ will be defined as

$$J = \int_0^T G(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi})dt \tag{2 3}$$

where the integration is carried over the path traversed by $\mathbf{X}$ and a terminal payoff as the magnitude of $F$ at the point where $\mathbf{X}$ intersects $C$

Let $V(\mathbf{X})$ be the optimal value of the payoff as accumulated from an initial position $\mathbf{X}$, and let the function $H$ be defined as§

$$H(\mathbf{V}',\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi}) = \sum_{j=1}^{n+1} \frac{\partial V}{\partial x_j} f_j(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi}) + G(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi}) \tag{2 4}$$

if the payoff is integral, or

$$H(\mathbf{V}',\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi}) = \sum_{j=1}^{n+1} \frac{\partial V}{\partial x_j} f_j(\mathbf{X},\boldsymbol{\phi},\boldsymbol{\psi})$$

if the payoff is terminal, where ¶

$$\mathbf{V}' \equiv \left(\frac{\partial V}{\partial x_i}, \quad, \frac{\partial V}{\partial x_{n+1}}\right)$$

Then it is shown in Ref 3 that if $\mathbf{X}^*$, $\boldsymbol{\phi}^*$, $\boldsymbol{\psi}^*$ constitute an optimal solution, i e, Eq (2-2) is satisfied along with a set of boundary conditions and $J$ or $F$ is minimized, then

$$H\{\mathbf{V}'[\mathbf{X}^*(t)], \mathbf{X}^*(t)\ \boldsymbol{\phi}^*(t),\boldsymbol{\psi}^*(t)\} = \boldsymbol{\phi}(\min)\boldsymbol{\psi}(\max)H\{\mathbf{V}'[\mathbf{X}^*(t)], \mathbf{X}^*(t),\boldsymbol{\phi},\boldsymbol{\psi}\} \tag{2 5}$$

at each instant of time, $0 \leq t \leq T$, where the min max is to be taken over all admissible values of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ In addition,

$$H[\mathbf{V}'(t),\mathbf{X}^*(t),\boldsymbol{\phi}^*(t),\boldsymbol{\psi}^*(t)] \equiv 0 \tag{2 6}$$

If Eq (2 6) is differentiated with respect to $\tau$, where $\tau = T - t$, if the optimal strategies are substituted into the resulting expression, and if we denote $\partial V/\partial x_i[\mathbf{X}^*(T - t)]$ by $V_i(\tau)$, then the following ensues:

$$\frac{d}{d\tau}\left(\frac{\partial V}{\partial x_i}\right) = \dot{V}_i = \sum_{j=1}^{n+1} V_j \frac{\partial f_j}{\partial x_i} [\mathbf{X}^*(t),\boldsymbol{\phi}^*(t),\boldsymbol{\psi}^*(t)] + \frac{\partial G}{\partial x_i} [\mathbf{X}^*(t),\boldsymbol{\phi}^*(t),\boldsymbol{\psi}^*(t)] \qquad i = 1,2, \quad, n+1 \tag{2 7}$$

where we merely suppress the $G$ when considering a terminal payoff Eqs (2 5) and (2 7) give rise to $n + 1 + r + s$ equations in as many unknowns, i e, $\phi_i^*,\psi_j^*,V_k$

In order to specify completely the optimal strategies and subsequently the equations of motion, $n$ boundary conditions must be determined Let us assume that $k < n$ of these conditions are specified at $\tau = 0$ and $\tau = T$, i e, $x_i = c_i$ $(i = 1,2, \quad, k)$, and let the terminal surface $C$ be represented in parametric form by $x_i = x_i\ (S_1, S_2, \quad, S_{n\ k\ 1})$ $(i = k + 1, \quad, n)$ Noting that $V = 0$ or $V = F$ on $C$ according to whether the payoff of the game is integral or terminal, allows $n$-$k$ 1 of the remaining $n$ $k$ constants to be determined from

$$\frac{\partial V(\mathbf{S})}{\partial S_m} = \sum_{i=k+1}^{n+1} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial S_m} \qquad m = 1,2, \quad, n\text{-}k\ 1 \tag{2 8}$$

‡ The vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are defined in regions of Euclidean space of dimensions $r$ and $s$, respectively, and are assumed to be piecewise continuous

§ Although $V(\mathbf{X})$ will be assumed, in what follows, to be both differentiable and possess continuous derivatives, this does not constitute a formal mathematical restriction on the optimality equations, since equivalent results could also be derived by employing adjoint variables

¶ These equations were labeled the main equation in Ref 3 If the duration of the game is not fixed and if $f_j$ and $G$ do not depend on $x_{n+1}$, then it can be shown that $\partial V/\partial x_{n+1} = 0$

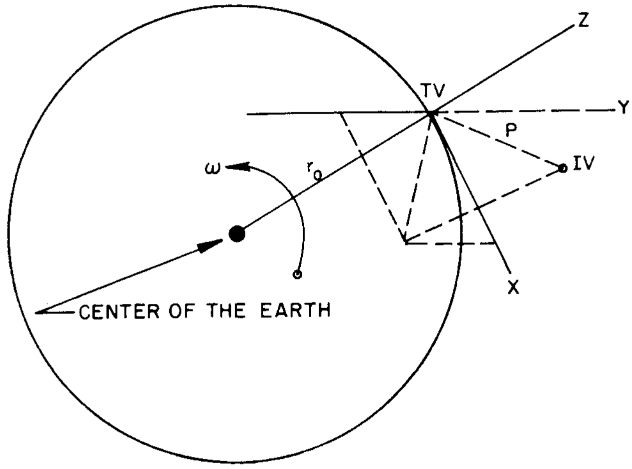


**Fig 1 The orbital coordinate system**

The main equation (2 4) supplies the final expression required for the complete determination of the unspecified constants

It is worth noting that problems that fall into the category of ordinary optimization, i e , one-sided games, can be handled by simply omitting wherever they appear, any reference to a second vehicle

As stated previously, this section has presented only a very brief summary of the philosophy of differential games and is intended to serve merely as a reference to the required optimality equations The reader is once again directed to the references for a complete explanation (or to any text treating the maximum principle)

## III Analysis

### A Passive Rendezvous

In order to initiate the analysis, consider a target vehicle ($TV$) traveling in a circular orbit of radius $r_0$ about the earth Let the triple $x,y,z$ represent the coordinate axes of a right-handed Cartesian coordinate system centered at $TV$ with $z$ continually directed radially outward from the center of the earth and $x$ tangent to the orbital track The remaining coordinate $y$ is normal to this plane of motion Figure 1 illustrates the geometry schematically The direction of motion of $TV$, as noted, is counter-clockwise Let a second interceptor vehicle ($IV$), to which we will attribute a variable acceleration engine, be in a nearby orbit To couch this latter statement more precisely, it will be presumed that the relative distance $P(t)$ between $IV$ and $TV$ is small compared to $r_0$ for all $t$ That is,

$$P(t) = (x^2 + y^2 + z^2)^{1/2} \ll r_0 \tag{3 1}$$

where $x,y,z$ now stand for the coordinates of the interceptor Under the assumptions noted in the introduction and with the added constraint implied by Eq (3 1), the linearized equations of motion of the interceptor relative to the rotating $x,y,z$ reference frame are

$$\frac{d^2x}{dt^2} = 2\omega \frac{dz}{dt} + \alpha_x \tag{3 2}$$

$$\frac{d^2z}{dt^2} = -2\omega \frac{dx}{dt} + 3\omega^2 z + \alpha_z \tag{3 3}$$

$$\frac{d^2y}{dt^2} = -\omega^2 y + \alpha_y \tag{3 4}$$

where $t$ is the time, $\omega$ is the angular velocity of $TV$ about the earth, and $\alpha_x$, $\alpha_y$, $\alpha_z$ are the components of the acceleration vector applied by $IV$ These equations have been derived by various authors (see Refs 5 and 6) and we therefore merely state their validity For a detailed derivation the reader is directed to the references The direction of the acceleration vector $\boldsymbol{\alpha}$ is assumed unconstrained The following transformation of coordinates is introduced:

$$\left.\begin{aligned} x_1 &= x & x_2 &= dx/dt \\ x_3 &= z & x_4 &= dz/dt \\ x_5 &= y & x_6 &= dy/dt \\ x_7 &= t & & \\ \alpha_1 &= \alpha_x & \alpha_2 &= \alpha \\ \alpha_3 &= \alpha_y & & \end{aligned}\right\} \tag{3 5}$$

In terms of the coordinates $x_i (i = 1,2, \quad , 7)$ the equations of motion of $IV$ now become

$$dx_1/dt = x_2 \tag{3 6}$$

$$dx_2/dt = 2\omega x_4 + \alpha_1 \tag{3 7}$$

$$dx_3/dt = x_4 \tag{3 8}$$

$$dx_4/dt = -2\omega x_2 + 3\omega^2 x_3 + \alpha_2 \tag{3 9}$$

$$dx_5/dt = x_6 \tag{3 10}$$

$$dx_6/dt = -\omega^2 x_5 + \alpha_3 \tag{3 11}$$

$$dx_7/dt = 1 \tag{3 12}$$

Equations (3 6–3 12) are a set of linear first-order autonomous differential equations whose solution for given values of $\alpha_1$, $\alpha_2$, and $\alpha_3$ completely determine a path traversed by $IV$ The state variables $x_1, \quad , x_7$ are, in terms of the notation formerly introduced, the descriptive variables of the interceptor, whereas $\alpha_1$, $\alpha_2$, and $\alpha_3$ are the navigation variables

The problem under consideration in this section is that of transferring the interceptor from a given initial state $x_i(0)$ $(i = 1\ 2, \quad , 7)$ to a given final state $x_i(T)$ $(i = 1,2, \quad , 7)$ with the minimum expenditure of fuel In particular, the final state of $IV$ will be taken such that the interceptor and target vehicles are copunctual both in position and velocity space; that is, the descriptive variables of both vehicles will be identical (Fig 2) The optimum controls and the trajectory flown are desired as outputs of the analysis

It is readily perceived that minimizing the propellant expended is equivalent to both maximizing the payload and minimizing the energy Reference 7 (among others) has shown that minimizing the fuel is equivalent to minimizing the following integral**:

$$J = \frac{1}{2} \int_0^T (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) dt \tag{3 13}$$

Inasmuch as the payoff of our single-sided game is integral and since the transfer time is prescribed, the main equation, according to our previous discussion, emerges in the following form:

$$\begin{aligned} H = {} & V_1 x_1 + V_2(2\omega x_4 + \alpha_1) + V_3 x_4 + \\ & V_4(-2\omega x_2 + 3\omega^2 x_3 + \alpha_2) + V_5 x_6 + \\ & V_6(-\omega^2 x_5 + \alpha_3) + V_7 + \tfrac{1}{2}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \end{aligned} \tag{3 14}$$

** The actual amount of propellant used is given by the expression

$$W_0 \left\{ 1 - \frac{1}{1 + \dfrac{W_0 a}{2W_W} \displaystyle\int_0^T (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\, dt} \right\}$$

where $(1/a)$ is the specific power, $W_0$ is the initial weight of the vehicle, and $W_W$ is the weight of the power supply To avoid ambiguity, all comments will allude to Eq (3 13) as yielding the fuel requirement

It is desired to select the navigation variable $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$ such that $H[\mathbf{V}'(t),\mathbf{X}(t),\boldsymbol{\alpha}^*]$ is a minimum Because the $\alpha_i(i = 1,2,3)$ are not restricted in magnitude, this can be accomplished by simply equating the several partial derivatives of $H$ with respect to the $\alpha_i$ to zero The acceleration program subsequently procured is

$$\alpha_1 = -V_2 \qquad \alpha_2 = -V_4 \qquad \alpha_3 = -V_6 \quad (3\ 15)$$

where the starred superscript has been dropped for purposes of clarity Holding the foregoing in abeyance, let us examine the differential equations for the $V_i$ From Eqs (2 7) and (3 6–3 12), the following is immediately obtained:

$$\dot{V}_1 = 0 \quad (3\ 16)$$

$$\dot{V}_2 = V_1 - 2\omega\ V_4 \quad (3\ 17)$$

$$\dot{V}_3 = 3\omega^2\ V_4 \quad (3\ 18)$$

$$\dot{V}_4 = V_3 + 2\omega\ V_2 \quad (3\ 19)$$

$$\dot{V}_5 = -\omega^2\ V_6 \quad (3\ 20)$$

$$\dot{V}_6 = V_5 \quad (3\ 21)$$

$$\dot{V}_7 = 0 \quad (3\ 22)$$

where, as will be recalled, the notation ( ) signifies differentiation with respect to $\tau$ Equations (3 16) and (3 22) yield

$$V_1 = V_1{}^0 = a_1 \qquad V_7 = V_7{}^0 \quad (3\ 23)$$

Differentiating Eq (3 19) and then using Eqs (3 17, 3 18, and 3 22) leads to the following differential equation for $V_4$:

$$\ddot{V}_4 + \omega^2 V_4 = 2\omega a_1 \quad (3\ 24)$$

The solution to Eq (3 24) is immediately

$$V_4 = a_2 \cos\omega\tau + a_3 \sin\omega\tau + (2/\omega)a_1 \quad (3\ 25)$$

where $a_1$, $a_2$, and $a_3$ are constants Similarly, the explicit expressions for the remaining $V_i$ can readily be demonstrated to be

$$V_2 = -3a_1\tau - 2a_2 \sin\omega\tau + 2a_3 \cos\omega\tau + a_4 \quad (3\ 26)$$

$$V_3 = 6\omega a_1\tau + 3\omega a_2 \sin\omega\tau - 3\omega a_3 \cos\omega\tau - 2\omega a_4 \quad (3\ 27)$$

$$V_5 = -\omega a_5 \sin\omega\tau + \omega a_6 \cos\omega\tau \quad (3\ 28)$$

$$V_6 = a_5 \cos\omega\tau + a_6 \sin\omega\tau \quad (3\ 29)$$

Although Eqs (3 27) and (3 28) are of no immediate importance, they will find application in the sections to follow Equations (3 25) and (3 29) indicate that the components of the optimum acceleration vector applied to the interceptor in the $z$ and $y$ directions are periodic, the frequency of the functions being exactly equal to the frequency of motion of the target vehicle about the earth The fact that $\alpha_3(= \alpha_y)$ exhibits this mode is not at all surprising, since the motion of $IV$ relative to $TV$ in the $y$ direction is also very nearly sinusoidal with the frequency referred to previously Equation (3 26), on the other hand, implies that the remaining component $\alpha_1(= \alpha_x)$ contains a secular term, i e, if $a_1 \neq 0$ Equations (3 25, 3 26, and 3 29) define the magnitude and direction of $\boldsymbol{\alpha}$ The equations of motion of $IV$ can now be determined Letting $t = T - \tau$ in Eq (3 9) and then differentiating the resulting expression with respect to this latter variable gives

$$\ddot{x}_4 = 2\omega\dot{x}_2 - 3\omega^2\dot{x}_3 - \dot{\alpha}_2 \qquad = -\omega^2 x_4 - 2\omega\alpha_1 - \dot{\alpha}_2 \quad (3\ 30)$$

Noting that $\alpha_1 = -V_2$ and $\alpha_2 = -V_4$, allows Eq (3 30) to be expressed as

$$\ddot{x}_4 + \omega^2 x_4 = -6\omega a_1\tau - 5\omega a_2 \sin\omega\tau + 5a_3\omega \cos\tau + 2\omega a_4 \quad (3\ 31)$$

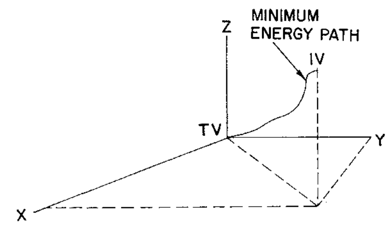


**Fig 2 Schematic of a passive rendezvous maneuver**

The solution to the foregoing is

$$x_4 = -\frac{6a_1\tau}{\omega} + \left(\frac{10a_2\omega\tau + 5a_3 + 4\omega b_1}{4\omega}\right)\cos\omega\tau + \left(\frac{-5a_2 + 10a_3\omega\tau + 4\omega b_2}{4\omega}\right)\sin\omega\tau + \frac{2a_4}{\omega} \quad (3\ 32)$$

Using straightforward procedures, the expressions for the remaining descriptive variables can be ascertained †† Specifically,

$$x_1 = -\frac{3a_1\tau^3}{2} + \frac{3a_1\tau^2}{2} - b_4\tau - \left(\frac{10a_2\omega\tau + 21a_3 + 4\omega b_1}{2\omega^2}\right)\cos\omega\tau - \left(\frac{-21a_2 + 10a_3\omega\tau + 4\omega b_2}{2\omega^2}\right)\sin\omega\tau + b_3 \quad (3\ 33)$$

$$x_2 = \frac{9a_1\tau^2}{2} - 3a_4\tau + \left(\frac{-11a_2 + 10a_3\omega\tau + 4\omega b_2}{2\omega}\right)\cos\omega\tau - \left(\frac{10a_2\omega\tau + 11a_3 + 4\omega b_1}{2\omega}\right)\sin\omega\tau + b_4 \quad (3\ 34)$$

$$x_3 = \frac{3a_1\tau^2}{\omega} - \frac{2a_4\tau}{\omega} + \left(\frac{-15a_2 + 10a_3\omega\tau + 4\omega b_2}{4\omega^2}\right)\cos\omega\tau - \left(\frac{10a_2\omega\tau + 15a_3 + 4\omega b_1}{4\omega^2}\right)\sin\omega\tau + \frac{2\omega b_4 + (8/\omega)a_1}{3\omega^2} \quad (3\ 35)$$

$$x_5 = (1/4\omega^2)(-a_5 + 2a_6\omega\tau + 4\omega^2 b_5)\cos\omega\tau - (1/4\omega^2)(2a_5\omega\tau + a_6 - 4\omega^2 b_6)\sin\omega\tau \quad (3\ 36)$$

$$x_6 = (1/4\omega)(2a_5\omega\tau - a_6 - 4\omega^2 b_6)\cos\omega\tau + (1/4\omega)(a_5 + 2a_6\omega\tau + 4\omega^2 b_5)\sin\omega\tau \quad (3\ 37)$$

where the $b_i(i = 1,2, \quad , 6)$ are constants of integration Because of the nature of $\alpha_3(= \alpha_y)$, only the motion of the interceptor in the $y$ direction is devoid of pure secular terms [see Eqs (3 36) and (3 37)] Since there are 12 constants to be evaluated and 6 equations with prescribed end points, the constants can be determined by solving a set of 12 linear homogeneous expressions In particular, when $\tau = 0$, i e, $t = T$, $x_i = x_i{}^0 = 0$ $(i = 1,2, \quad , 6)$ these expressions re-

†† See Refs 8–11 for discussions of the fixed time-minimum energy problem, with general linear differential accessory conditions

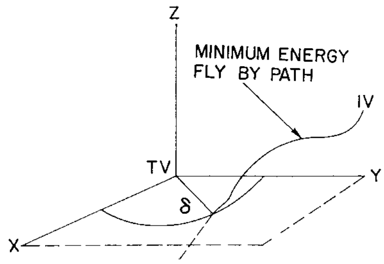


**Fig 3 Schematic of a passive fly-by maneuver**

duce to‡‡

$$21a_3 + 4\omega b_1 - 2\omega^2 b_3 = 0 \qquad (3\ 38)$$

$$-11a_2 + 4\omega b_2 + 2\omega b_4 = 0 \qquad (3\ 39)$$

$$32a_1 - 45\omega a_2 + 12\omega^2 b_2 + 8\omega^2 b_4 = 0 \qquad (3\ 40)$$

$$5a_3 + 8a_4 + 4\omega b_1 = 0 \qquad (3\ 41)$$

$$-a_5 + 4\omega^2 b_5 = 0 \qquad (3\ 42)$$

$$a_6 + 4\omega^2 b_6 = 0 \qquad (3\ 43)$$

Employing Eqs (3 42) and (3 43) in conjunction with Eqs (3 36) and (3 37) allows $b_5$ and $b_6$ to be expressed as

$$b_5 = \frac{x_6(0)\{\omega T \cos\omega T - \sin\omega T\} - x_5(0)\omega^2 T \sin\omega T}{2\omega(\omega^2 T^2 - \sin^2\omega T)} \qquad \omega T \neq 0 \quad (3\ 44)$$

$$b_6 = -\frac{x_6(0)\omega T \sin\omega T + x_5(0)\omega\{\omega T \cos\omega T + \sin\omega T\}}{2\omega(\omega^2 T^2 - \sin^2\omega T)} \qquad \omega T \neq 0 \quad (3\ 45)$$

and therefore

$$a_5 = 2\omega\left[\frac{x_6(0)\{\omega T \cos\omega T - \sin\omega T\} - x_5(0)\omega^2 T \sin\omega T}{\omega^2 T^2 - \sin^2\omega T}\right] \qquad \omega T \neq 0 \quad (3\ 46)$$

$$a_6 = 2\omega\left[\frac{x_6(0)\omega T \sin\omega T + x_5(0)\omega\{\omega T \cos\omega T + \sin\omega T\}}{\omega^2 T^2 - \sin^2\omega T}\right] \qquad \omega T \neq 0 \quad (3\ 47)$$

The remaining eight constants involve extremely complicated expressions which, if noted, would add little to this exposition The necessary expressions can be easily put in determinant form for systematic evaluation by a computer §§

The sole quantity of interest yet to be determined explicitly is the indicator of fuel consumption $J$ Substituting Eqs (3 26, 3 28, and 3 29) into Eq (3 13) for $\alpha_1$, $\alpha_2$, and $\alpha_3$ and integrating allows $J$ to be expressed in the following form:

$$J = \frac{1}{2}\left\{3a_1^2\tau^3 - 3a_1a_4\tau^2 + \frac{(8a_1^2 + 5a_2^2\omega^2 + 5a_3^2\omega^2 + 2a_4^2\omega^2 + a_5^2\omega^2 + a_6^2\omega^2)\tau}{2\omega^2} - \frac{(-3a_2a_3 + a_5a_6)}{2\omega}\cos 2\omega\tau + \frac{(-3a_2^2 + 3a_3^2 + a_5^2 - a_6^2)}{4\omega}\sin 2\omega\tau + \frac{4(a_2a_4\omega - 4a_1a_3 - 3a_1a_4\omega\tau)}{\omega^2}\cos\omega\tau + \frac{4(4a_1a_2 + a_3a_4\omega - 3a_1a_3\omega\tau)}{\omega^2}\sin\omega\tau\right\}_0^T \qquad (3\ 48)$$

‡‡ It should be recognized that Eqs (3 31–3 37) apply without modification if the $x_i^0$ are taken to be something other than zero That is, the equations are more general than the specific application to which they are applied here

§§ Rather formidable 8 × 8 determinants are generally involevd

### B Passive Fly-By

The problem under consideration in this section deals with what has been previously described as passive fly by Briefly, it is desired to obtain the optimal acceleration program and the subsequent trajectory that will allow the interceptor to be located at a distance $\delta$, measured in the $x$-$y$ plane, from the target at a specified time $T$ (Fig 3) As in the previous section, we will leave the direction and magnitude of acceleration vector to be determined by the optimization

In light of what has already been derived in Sec III A, the solution here is almost immediate In particular all of the equations of Sec III A up to and including Eq (3 37) and the equation for $J$ apply without modification to the problem in hand Only the constants that appear in these equations must be re-evaluated Because both the terminal velocity and one of the terminal coordinates have not been specified a priori, additional equations are needed to supply the necessary expressions for evaluating these constants; that is, four subsidiary equations are required These equations can be obtained by parameterizing the coordinates on the terminal surface (i e , $x_3^0 = 0$, $x_7^0 = T, x_1^{0^2} + x_5^{0^2} = \delta^2$) and then employing Eq (2 8) Specifically, on the surface $C$, the descriptive variables can be represented in the following parameterized form:

$$\begin{aligned} x_1^0 &= S_1 \\ x_2^0 &= S_2 \\ x_3^0 &= 0 \\ x_4^0 &= S_4 \\ x_5^0 &= S_5 = \pm(\delta^2 - S_1^2)^{1/2} \\ x_6^0 &= S_6 \\ x_7^0 &= T \end{aligned} \qquad (3\ 49)$$

Using Eqs (3 49) in (2 8) and noting that $V = 0$ on $C$ (since the payoff is integral), we obtain the following relationships:

$$V_2^0 = V_4^0 = V_6^0 = 0 \qquad (3\ 50)$$

$$V_1^0 = \frac{V_5^0 S_1}{\pm(\delta^2 - S_1^2)^{1/2}} \qquad (3\ 51)$$

Equation (3 50) in conjunction with Eq (3 15) indicates that the modulus of the acceleration vector ($\alpha$) is zero on the terminal surface In addition, Eqs (3 50) and (3 51) supply the expressions required to determine the remaining constants appearing in the path equations Evaluation of Eqs (3 25, 3 26, and 3 29) on $C$ leads to

$$a_1 = -(\omega a_2/2) \qquad (3\ 52)$$

$$a_4 = -2a_3 \qquad (3\ 53)$$

$$a_5 = 0 \qquad (3\ 54)$$

Since, by hypothesis, at $\tau = 0$, $x_3^0 = 0$, Eqs (3 35) and (3 52)

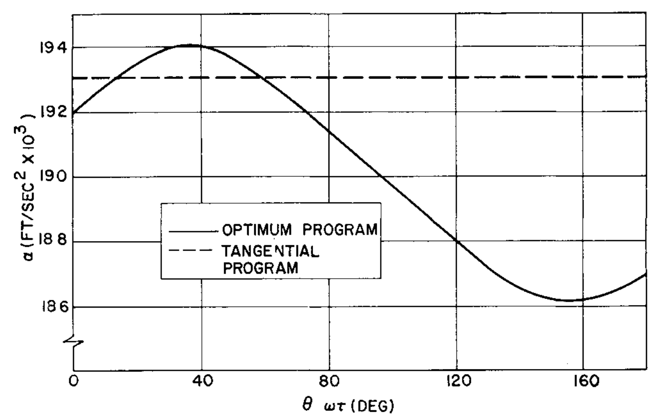


**Fig 4 Acceleration programs**

when solved for $a_2$ and $a_1$ yield

$$a_2 = (4\omega/61)(3b_2 + 2b_4) \qquad (3\ 55)$$

$$a_1 = -(2\omega^2/61)(3b_2 + 2b_4) \qquad (3\ 56)$$

Furthermore, Eqs (3 33, 3 36, and 3 28) on $C$ take the form

$$x_1{}^0 = S_1 = -\frac{21a_3 + 4\omega b_1 - 2\omega^2 b_3}{2\omega^2} \qquad (3\ 57)$$

$$x_5{}^0 = \pm(\delta^2 - S_1{}^2)^{1/2} = b_5 \qquad (3\ 58)$$

$$V_5{}^0 = \omega a_6 \qquad (3\ 59)$$

Substituting Eq (3 57) into (3 58) and solving for $a_3$ in terms of the $b_i$ gives¶¶

$$a_3 = -\frac{\pm 2\omega^2(\delta^2 - b_5{}^2)^{1/2} + 4\omega b_1 + 2\omega^2 b_3}{21} \qquad (3\ 60)$$

From Eq (3 53), the constant $a_4$ becomes

$$a_4 = \frac{\pm 4\omega\ (\delta^2 - b_5{}^2)^{1/2} - 8\omega b_1 + 4\omega^2 b_3}{21} \qquad (3\ 61)$$

Finally, substituting Eqs (3 58, 3 59, and 3 56) into Eq (3 51) allows $a_6$ to be expressed as

$$a_6 = -\left(\frac{2\omega}{61}\right)\frac{b_5}{\pm(\delta^2 - b_5{}^2)^{1/2}}(3b_2 + 2b_4) \qquad (3\ 62)$$

Equations (3 54–3 56 and 3 60–3 62) give the $a_i$ as functions of the constants $b_i (i = 1,2, \quad , 5)$ If these expressions are subsequently employed in the path equations, Eqs (3 23–3 37) then there evolves a set of 6 nonlinear algebraic expressions which are to be solved at the initial conditions $x_i(0)$ for the constants $b_i (i = 1,2, \quad 6)$ Once these constants have been obtained, the solution is completely determined It is worth noting that only one of the constants, namely $b_5$, appears nonlinearly The path equations can thus be solved at the boundary conditions for the remaining constants in terms of $b_5$ This should expedite the solution process

Although the evaluation of these constants by hand would involve a prohibitive amount of time (if it could actually be accomplished at all), machine routines do exist which can perform the necessary operations A successful determination of the $b_i$ by a computer indeed may also involve a con siderable amount of time, depending on the convergence properties of the routines used Be this as it may, the numerical problems are left to the user

### C Active Rendezvous

This section of the report deals with a problem of pursuit and evasion It is included herein mainly to illustrate the full application of the differential games technique Be cause realistic problems which fall into the category of games, are generally complicated and often involve unwieldy equations, our concern here will be mainly to obtain the functional form of the optimum navigation variables Once this has been accomplished, the differential equations of motion of the vehicles can (theoretically) be integrated to obtain the path traversed by the vehicles

Consider two vehicles $P$ and $E$ each equipped with low acceleration engines The magnitudes of the acceleration vectors developed by these engines are presumed to be constant while their directions are free The pursuit vehicle $P$ desires to rendezvous with $E$ in the minimum time, whereas conversely, the evader $E$ wishes to maximize this time The term rendezvous is used here to mean coincidence in both position and velocity space ***

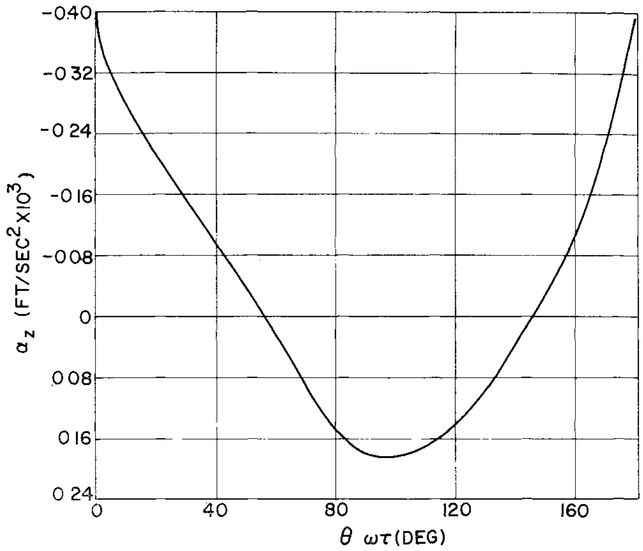


**Fig 5 Outward component of optimum acceleration program**

Under the assumptions used in the preceding sections, the main equation for an integral payoff with $G = 1$ is easily seen to be†††

$$H = \sum_{J=1}^{2}\{V_{1_j}x_{2_j} + V_{2_j}(2\omega x_{4_j} + \alpha_j \cos\theta_j \sin\phi_j) + V_{3_j}x_{4_j} + V_{4_j}(-2\omega x_{2_j} + 3\omega^2 x_{3_j} + \alpha_j \sin\theta_j) + V_{5_j}x_{6_j}(-\omega^2 x_{5_j} + \alpha_j \cos\theta_j \cos\phi_j) + 1\} = 0 \qquad (3\ 63)$$

where the subscripts $j = 1$ and 2 allude to the pursuer and evader, respectively Here, as noted, it is more convenient to represent the acceleration vector of the vehicle ($\boldsymbol{\alpha}_j$) in terms of spherical coordinates Performing the necessary min max operation described in Sec II leads to the following:

$$V_{4_j}\cos\theta_j - (V_{2_j}\sin\phi_j + V_{6_j}\cos\phi_j)\sin\theta_j = 0 \qquad j = 1,2 \qquad (3\ 64)$$

$$V_{2_j}\cos\phi_j - V_{6_j}\sin\phi_j = 0 \qquad j = 1,2 \qquad (3\ 65)$$

Solving the foregoing for $\theta_j$ and $\phi_j$ yields

$$\sin\theta_j = \frac{AV_{4_j}}{(V_{2_j}{}^2 + V_{4_j}{}^2 + V_{6_j}{}^2)^{1/2}} \qquad (3\ 66)$$

$$\cos\theta_j = \frac{(V_{2_j}{}^2 + V_{6_j}{}^2)^{1/2}}{(V_{2_j}{}^2 + V_{4_j}{}^2 + V_{6_j}{}^2)^{1/2}} \qquad (3\ 67)$$

$$\sin\phi_j = \frac{AV_{2_j}}{(V_{2_j}{}^2 + V_{6_j}{}^2)^{1/2}} \qquad A = \begin{cases} -1 \text{ if } j = 1 \\ \ \ 1 \text{ if } j = 2 \end{cases} \qquad (3\ 68)$$

$$\cos\phi_j = \frac{AV_{6_j}}{(V_{2_j}{}^2 + V_{6_j}{}^2)^{1/2}}$$

If Eqs (3 25, 3 26, and 3 29) are substituted into Eqs (3 67) and (3 68), then the navigation variables are known in terms of the constants $a_{ij} (i = 1,2, \quad , 6, j = 1,2)$

¶¶ Note that there are two possible values for $a_3$ and subse quently for $a_4$ and $a_6$ The correct choice of sign for the radical $(\delta^2 - b_5{}^2)^{1/2}$ can most easily be obtained by computing the corresponding values of $J$

*** There is, of course, a possibility that rendezvous will not occur for any finite time However, we will assume the contrary

††† In this case, the origin of the $x,y,z$ coordinate system remains in some arbitrary, nearby circular orbit and rotates with a constant angular velocity; that is, it is not fixed in the evader

The terminal surface $C$, in parametric form, is

$$x_{1_i}{}^0 = S_1$$
$$x_{2_i}{}^0 = S_2$$
$$\vdots \qquad (3\ 69)$$
$$x_{6_i}{}^0 = S_6$$

Equation (2 8) in conjunction with Eq (3 69) leads to the following expressions relating the $V_{ij}{}^0$:

$$V_{i_1}{}^0 = -V_{i_2}{}^0 \qquad i = 1,2, \quad , 6 \qquad (3\ 70)$$

Solving Eq (3 70) [i e , Eqs (3 23 and 3 25–3 29)] leads to the result

$$a_{i_1} = -a_{i_2} \qquad i = 1,2, \quad , 6 \qquad (3\ 71)$$

and hence

$$V_{i_1} = -V_{i_2} \qquad (3\ 72)$$

at all times Substitution of Eq (3 72) into Eqs (3 67) and (3 68) gives the interesting result‡‡‡

$$\sin\theta_1 = \sin\theta_2 \qquad \cos\theta_1 = \cos\theta_2$$
$$\sin\phi_1 = \sin\phi_2 \qquad \cos\phi_1 = \cos\phi_2 \qquad (3\ 73)$$

Equation (3 73) shows that the direction of the acceleration vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are identical at all times §§§

### D An Illustrative Example¶¶¶

Consider a vehicle traveling in a circular orbit at an altitude of 1000 naut miles above the earth Under a constant tangential acceleration program of magnitude $\alpha = 19\ 3067 \times 10^{-3}$ ft/sec², i e , $10^{-3}$ $g$'s, this vehicle will exhibit the following coordinates (as referred to the rotating $x,y,z$ system) at a time equal to one-half the period of the nominal orbit:

$$x_1{}^0 = 0\ 18376 \times 10^6 \text{ ft}$$
$$x_2{}^0 = 0\ 21525 \times 10^3 \text{ fps}$$
$$x_3{}^0 = 0\ 16905 \times 10^6 \text{ ft}$$
$$x_4{}^0 = 0\ 90119 \times 10^2 \text{ fps}$$

The propellant used in performing this change of state is

$$J = \tfrac{1}{2}[(19\ 3067)^2 10^{-6}]\ 3\ 7152 \times 10^3 = 0\ 6924 \text{ ft}^2/\text{sec}^3$$

Using the values of $x_1{}^0$ to $x_4{}^0$ just noted as terminal conditions and $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$ as initial conditions Eqs (3 32–3 35 and 3 48) can be used to obtain the equivalent optimum trajectory and minimal value of $J$ Carrying out the necessary calculations leads to the following result:

$$J_{\min} = \frac{1}{2}\int_0^T (\alpha_x{}^2 + \alpha^2)dt = 0\ 6718 \text{ ft}^2/\text{sec}^3$$

Comparing $J_{\min}$ to $J$ indicates approximately a 3% saving in mass using the optimum program The relatively small improvement of the optimum solution compared to the tangential program is not surprising in view of the inherent efficiency of tangential acceleration; that is, tangential acceleration maximizes the instantaneous rate of change of total energy

Figure 4 shows a plot of $|\boldsymbol{\alpha}_{\text{opt}}|$ and $\alpha$ vs central angle It is interesting to note that although $|\boldsymbol{\alpha}_{\text{opt}}|$ does not change appreciably over the duration of the mission, it is not constant Figure 5 displays the variation of $\alpha$ with time

A brief study was also conducted to determine the sensitivity of the value of $J_{\text{opt}}$ to small variations in the terminal conditions Some typical results obtained are indicated below:

$$\Delta x^0 = 0\ 1 \text{ fps} \rightarrow \Delta J = -0\ 0087 \text{ ft}^2/\text{sec}^3$$
$$\Delta x_3{}^0 = 100 \text{ ft} \rightarrow \Delta J = 0\ 0032 \text{ ft}^2/\text{sec}^3$$

### E Discussion

In the preceding analysis, solutions to a few optimum continuous thrust rendezvous problems have been presented Specifically, the passive rendezvous and passive fly-by maneuvers that minimize the required propellant have been obtained in closed form That is, the acceleration programs and related trajectories have been obtained as explicit functions of the time The problem of minimum time active rendezvous has also been discussed and the following general result was deduced, via the technique of differential games: the optimum strategies for pursuer and evader dictates that their acceleration vectors should be parallel at all times In addition to the foregoing, an illustrative example dealing with a modification of the passive rendezvous maneuver was presented In this example, an optimum acceleration program was compared with a constant tangential acceleration program known to be capable of performing the same change of state The optimum acceleration level was shown to be variable, and this program was slightly more efficient than the tangential program

‡‡‡ It has recently been brought to the author's attention that the results of this section are in agreement with those obtained in Ref 8, pp 226–237

§§§ Although not heretofore specifically indicated, it is obvious that the optimum strategy for the pursuer is dependent on the strategy adopted by the evader If the evader chooses not to employ the optimum strategy, then the pursuer will select an alternate that is optimal under these new conditions In Refs 1–4, the author uses some heuristic arguments to show that under these conditions a pursuer would rendezvous with an evader with less effort (or in less time) then if the evader had employed his optimum strategy

¶¶¶ The example to be considered involves an optimum change of state rather then a specific rendezvous [see footnote preceding Eq (3 38)]

## References

[1] Isaacs, R , "Differential games I: Introduction," Project Rand RM-1391 (November 30, 1954)

[2] Isaacs, R , "Differential games II: The definition and formulation," Project Rand RM-1399 (November 30, 1954)

[3] Isaacs, R , "Differential games III: The basic principles of the solution process," Project Rand RM-1411 (December 21, 1954)

[4] Isaacs, R , "Differential games IV: Mainly examples,' Project Rand RM-1486 (March 25, 1955)

[5] Wheelon, A , "An introduction to midcourse and terminal guidance," Space Technology Laboratories Tech Memo 145 (June 10, 1958)

[6] Hollister, W , "The design of a control system for the terminal phase of a satellite rendezvous," M S Thesis, Mass Inst Tech (June 1959)

[7] Irving, J H , "Low thrust flight: variable exhaust velocity in gravitational fields," *Space Technology*, edited by H S Seifert (John Wiley and Sons, Inc , New York, 1959), Chap 10

[8] Pontryagin, L S , Boltyanskii, V G , Gamkrelidze, R V , and Mischenko, E F , *The Mathematical Theory of Optimal Processes*, transl by K N Trirogoff, edited by L W Neustadt (Interscience Division, John Wiley and Sons, Inc , New York, 1962)

[9] Athanassiades, M , "Optimal control for linear time—invariant plants with time, fuel, and energy constraint," Trans Am Inst Elec Engrs App Ind 81, 321–325 (1963)

[10] Kalman R E , "The theory of optimal control and the calculus of variations," RIAS TR 61-3 (1961)

[11] Meditch, J S , "Synthesis of a class of linear minimum energy controls," Aerospace Corp A-63-1732 2-13 (April 1, 1963); also Inst Elec Electron Engrs Trans Auto Control (to be published)